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Lie Symmetry Analysis and Exact Solutions to the Quintic Nonlinear Beam Equation

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ABSTRACT

In this paper, the exact solutions to the equation of motion of the nonlinear vibration of Euler-Bernoulli beam which is governed by the quintic nonlinear equation are investigated by using Lie symmetry analysis. The leading tools for transforming the equation of motion which is in the form of partial differential equation into an ordinary differential equation are the infinitesimal generators. These generators are calculated by using technique of group transformation. The Lie algebra of the infinitesimal generator is spanned by four linearly independent generators. An optimal system of subalgebra is constructed. Invariants are calculated by solving the characteristic system and then designate one of invariant as a function of the others. Then the partial differential equation can be transformed to the ordinary differential equation. Based on an optimal system, in some cases the ordinary differential equation can be solved and exact solutions are obtained.

Keywords: Exact solutions; Euler-Bernoulli beam; Symmetry analysis; Lie group theory; Quintic non-linear beam equation.

1. Introduction

A wide variety of phenomena in natural science and technology can be formed according to the nonlinear partial differential equations. The exact solutions of these equations are usually difficult to find. However, the proper understandings of qualitative attributes of phenomena and the process in several areas are important to interest. The various methods such as stiffness analytical approximation method Sedighi et al. (2013),parameter expansion method Sedighi and Reza (2013),homotopy analysis method (HAM) Sedighi et al. (2012), homotopy perturbation method Sedighi et al. (2013), max-min Approach (MMA) Sedighi et al. (2013), iteration perturbation method (IPM) Sedighi et al. (2013), Lie and Noether symmetry analysis Liu and Li (2010), Bokhari et al. (2010), Chatanin et al. (2008) and Ozkaya and Pakdemirli (2002)have been developed to find the solutions of the partial differential equations.

Lie group theory is the mathematical discipline which is embodiment and synthesizer symmetry of differential equation Ibragimov (1996). Lie group theory is sometimes called Lie symmetry analysis. Exact solutions of the differential equations can be found from the solutions of the reduced equations. If the partial differential equation is invariant under a Lie group, the reduced equation will have fewer independent variables. After that, the partial differential equation can be transformed into ordinary differential equation for finding exact solutions. However the leading tools of Lie symmetry analysis which are used to reducing the number of independent variables are the infinitesimal generators. Furthermore, a basis of the Lie algebra is used to investigate the solvability of the equation. The effectiveness of Lie symmetry analysis has been demonstrated in the analysis of some beam problems such as Ozkaya and Pakdemirli (2002) and Bokhari et al. (2010).

A structural element of civil engineering which can be used to carry force and load in steel construction such as building, bridges and other is beam. At the first time, the exact solution of the beam problem was investigated, in the general term of elasticity equation, by Love (1927) (cited in Sedighi et al., 2013, p. 245). Afterward, the vibrating beam problems were considered. Beam problems were formulated in terms of the partial differential equation of motion with different boundary conditions for finding various solutions. In order to find the solutions to the equation of motion for the quintic nonlinear beam, many method were used. Homotopy analysis method was used to solve the transversal oscillation of quintic non-linear beam by Sedighi et al. (2012). The four methods such as stiffness analytical approximation method, homotopy perturbation method with an auxiliary term, max-min approach (MMA) and iteration perturbation method (IPM) were used to solve the governing equation of transversely vibrating quintic nonlinear beam by Sedighi et al. (2013). After that, Sedighi and Reza (2013) used parameter expansion method to perform the exact solution of beam vibration with quintic nonlinearity, including exact expressions for the beam curvature.

In this work, Lie symmetry analysis is applied for finding the exact solutions of the quintic nonlinear equation of motion. First of all the infinitesimal generators are determined and then the Lie algebra corresponding to the equation of motion is spanned by four linearly independent generators. Some cases of generators are used to obtain the exact solutions as invariant solutions.

2. Equation of Motion

The quintic nonlinear equation of motion for the nonlinear vibration of Euler-Bernoulli beam has been explained by Zhang et al. (2005). In this paper, the governing equation which was studied by Sedighi et al. (2012) is interested. A hinged-hinged flexible beam model with length l. subjected to constant axial load is shown in figure 1.

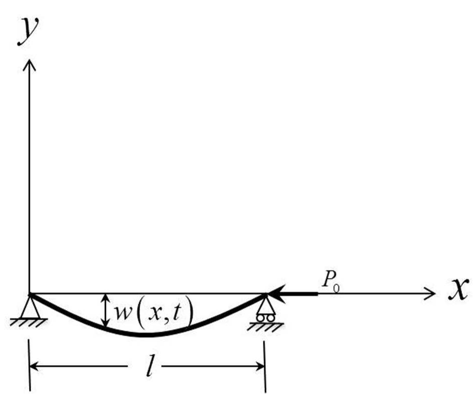

Figure 1: The model of a hinged-hinged flexible beam subjected to constant axial load (Sedighi, et al., 2012)

In the Cartesian coordinate system, the flexural deflection with respect to x, t is denoted by w. From Sedighi et al. (2012), in order to simplify the quintic nonlinear equation of motion, only the fundamental transvers mode is studied. The assumption for neglecting the interaction between transverse and longitudinal vibrations is considered. In the equation of motion, the important constants are a constant axial force P_0 , the damping coefficient c , mass per unit length m , a modulus of elasticity E and a moment of inertia I. The quintic nonlinear equation of motion is obtained as follows:

$$
\begin{aligned}
mw_{tt} + cw_t + EI\left[\frac{27}{2}w_x^2 w_{xx}^3 - 3w_{xx}^3 - 3w_x^2 w_{xxxx} + \frac{9}{4}w_x^4 w_{xxxx}\right] \\
+EIw_{xxxx} + P_0 w_{xx} + \frac{3}{2}P_0 w_x^2 w_{xx} = 0. \qquad (1)
\end{aligned}
$$

Next, the following dimensionless terms are introduced for reducing the number of constants

$$
\overline{x} = \frac{x}{t}, \bar{t} = \sqrt{\frac{EI}{ml^4}}, \overline{w} = \frac{w}{l}, \mu = \frac{l^2 c}{\sqrt{mEI}}, \beta = \frac{P_0 l^2}{EI}. \qquad (2)
$$

Therefore, the non-dimensional nonlinear equation of motion is obtained as follow:

$$
\begin{aligned}
\overline{w}_{\overline{tt}} + \mu\overline{w}_{\bar{t}} + \frac{27}{2}\overline{w}_{\overline{x}}^2\overline{w}_{\overline{xx}}^3 - 3\overline{w}_{\overline{xx}}^3 - 3\overline{w}_{\overline{x}}^2\overline{w}_{\overline{xxxx}} + \frac{9}{4}\overline{w}_{\overline{x}}^4\overline{w}_{\overline{xxxx}} \\
+\overline{w}_{\overline{xxxx}} + \beta\overline{w}_{\overline{xx}} + \frac{3}{2}\beta\overline{w}_{\overline{x}}^2\overline{w}_{\overline{xx}} = 0. \qquad (3)
\end{aligned}
$$

3. Calculation of infinitesimal generators

In this section, Lie symmetry analysis will be applied to the quintic nonlinear equation of motion. First of all, infinitesimal generators for the quintic nonlinear equation of motion are written in the following form

$$
V = \xi(\overline{x}, \bar{t}, \overline{w})\frac{\partial}{\partial\overline{x}} + \tau(\overline{x}, \bar{t}, \overline{w})\frac{\partial}{\partial\bar{t}} + \phi(\overline{x}, \bar{t}, \overline{w})\frac{\partial}{\partial\overline{w}} \qquad (4)
$$

where $\xi(\overline{x}, \bar{t}, \overline{w}), \tau(\overline{x}, \bar{t}, \overline{w})$ and $\phi(\overline{x}, \bar{t}, \overline{w})$ are coefficient functions.

The infinitesimal generator will be used to generate the symmetry group of (3). Since equation (3) is the fourth order partial differential equation, the fourth prolongation of the infinitesimal generator is needed, which is,

$$
\begin{aligned}
pr^{(4)}V &= pr^{(3)}V + \phi^{\overline{xxxx}}\frac{\partial}{\partial\overline{w}_{\overline{xxxx}}} + \phi^{\overline{xxxt}}\frac{\partial}{\partial\overline{w}_{\overline{xxxt}}} + \phi^{\overline{xxtt}}\frac{\partial}{\partial\overline{w}_{\overline{xxtt}}} \\
&\quad +\phi^{\overline{xttt}}\frac{\partial}{\partial\overline{w}_{\overline{xttt}}} + \phi^{\overline{tttt}}\frac{\partial}{\partial\overline{w}_{\overline{tttt}}} \qquad (5)
\end{aligned}
$$

After applying the fourth prolongation of V to the equation (3), all coefficient functions of the infinitesimal generator are obtained as follows:

$$\xi = k_1, \quad \tau = k_2, \quad \phi = k_3 e^{-\mu \bar{t}} + k_4. \tag{6}$$

So, the infinitesimal generators admitted by the quintic nonlinear equation of motion have the following form

$$V = k_1 \frac{\partial}{\partial \bar{x}} + k_2 \frac{\partial}{\partial \bar{t}} + \left[k_3 e^{-\mu \bar{t}} + k_4 \right] \frac{\partial}{\partial \bar{w}} \tag{7}$$

where k_1, k_2, k_3, k_4 are arbitrary constants.

The Lie algebra corresponding to the quintic nonlinear equation of motion is spanned by four linearly independent generators,

$$V_1 = \frac{\partial}{\partial \bar{x}}, V_2 = \frac{\partial}{\partial \bar{t}}, V_3 = \frac{\partial}{\partial \bar{w}}, V_4 = e^{-\mu \bar{t}} \frac{\partial}{\partial \bar{w}}. \tag{8}$$

After calculating the commutation relations between these four generators, we obtain

$[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = [V_4, V_4] = 0$
$[V_1, V_2] = [V_1, V_3] = [V_1, V_4] = [V_2, V_1] = [V_3, V_1] = [V_4, V_1] = 0$
$[V_2, V_3] = [V_3, V_2] = [V_3, V_4] = [V_4, V_3] = 0$
$[V_2, V_4] = -[V_4, V_2] = -\mu e^{-\mu \bar{t}} \frac{\partial}{\partial \bar{w}}$

It can be seen that, basis of the Lie algebra is closed under the Lie bracket.

By considering the adjoint representation, $Ad(exp(\varepsilon V_i))V_i = V_i$, $i = 1, 2, 3, 4$, we get $Ad(exp(\varepsilon V_i))V_1 = V_1$,$Ad(exp(\varepsilon V_i))V_3 = V_3$, $i = 1, 2, 3, 4$,
$Ad(exp(\varepsilon V_i))V_2 = V_2$, $i = 1, 2, 3$, $Ad(exp(\varepsilon V_i))V_4 = V_4$, $i = 1, 3, 4$,
$Ad(exp(\varepsilon V_2))V_4 = e^{-\mu\varepsilon} V_4$, $Ad(exp(\varepsilon V_4))V_2 = V_2 - \varepsilon \mu V_4$,
for any $\varepsilon \in R$.
Therefore, an optimal system of subalgebra is as follows:
$\{V_1 + V_2 + V_3, V_2 + V_3, V_1 + V_2, V_1 + V_3 + V_4, V_1 + V_4, V_1 + V_3, \}$,

4. Symmetry reduction and exact solutions

In this section, the nonlinear partial differential equation is reduced to the ordinary differential equations by using the infinitesimal generators based on the optimal system. The transformations are classified into four cases as shown in Table 1.

Table 1: Four cases of transformation.

Cases	Generators	Ordinary Differential Equations
1	$V_1 + V_2 + V_3$	$u' + \mu(u'+1) + \frac{27}{2}u'^2u''^3 - 3u''^3 - 3u'^2u^{(4)}$ $+\frac{9}{4}u'^4u^{(4)} + u^{(4)} + \beta u'' + \frac{3}{2}\beta u'^2u'' = 0$
2	$V_2 + V_3$	$\mu + \frac{27}{2}u'^2u''^3 - 3u''^3 - 3u'^2u^{(4)} + \frac{9}{4}u'^4u^{(4)}$ $+u^{(4)} + \beta u'' + \frac{3}{2}\beta u'^2u'' = 0$
3	$V_1 + V_2$	$u'' + \mu u' + \frac{27}{2}u'^2u''^3 - 3u''^3 - 3u'^2u^{(4)}$ $+\frac{9}{4}u'^4u^{(4)} + u^{(4)} + \beta u'' + \frac{3}{2}\beta u'^2u'' = 0$
4	$V_1 + V_3 + V_4$ $V_1 + V_4$ $V_1 + V_3$	$u'' + \mu u' = 0$

From table 1. $u' = \frac{du}{dz}$, z is described by $z = \bar{t} - \overline{x}$ in cases 1 and 3.
In case 2, z is described by $z = \overline{x}$.
In case 4, z is described by $z = \bar{t}$.

The first three cases, partial differential equation can be reduced to the ordinary differential equations. For the last case, the ordinary differential equations can be reduced to linear ordinary differential equation, which can be solved analytically. Next, the calculations for finding the exact solutions by using the generators in the last case are shown as follows:

(i) For $V_1+V_3+V_4 = \frac{\partial}{\partial \overline{x}} + \frac{\partial}{\partial \overline{w}} + e^{-\mu\bar{t}}\frac{\partial}{\partial \overline{w}}$, by solving the characteristic equation

$$\frac{d\overline{x}}{1} = \frac{d\overline{w}}{1+e^{-\mu\bar{t}}}, \tag{9}$$

we have $\overline{w} = u(z) + (1+e^{-\mu\bar{t}})\overline{x}$, where $z = \bar{t}$. By calculating their derivatives and substituting into equation (3), the equation of motion can be reduced to linear ordinary differential eqaution

$$u'' + \mu u' = 0, \tag{10}$$

where $u' = \frac{du}{dz}$.

Since equation (10) can be solved analytically, so the exact solution to equation (3) is obtained as follow:

$$\overline{w} = \overline{x}\left[(1+e^{-\mu\bar{t}})\right] + b_1e^{-\mu\bar{t}} + b_2, \tag{11}$$

where b_1, b_2 are arbitrary constants.

By the same way for (ii) $V_1 + V_4 = \frac{\partial}{\partial \overline{x}} + e^{-\mu \bar{t}} \frac{\partial}{\partial \overline{w}}$, we have $\overline{w} = u(z) + e^{-\mu \bar{t}} \overline{x}$, where $z = \bar{t}$. The equation of motion can be reduced to $u'' + \mu u' = 0$, where $u' = \frac{du}{dz}$ .Then the exact solution is obtained as follow:

$$\overline{w} = \overline{x} e^{-\mu \bar{t}} + b_1 e^{-\mu \bar{t}} + b_2, \tag{12}$$

where b_1, b_2 are arbitrary constants.

(iii) For $V_1 + V_3 = \frac{\partial}{\partial \overline{x}} + \frac{\partial}{\partial \overline{w}}$, we have $\overline{w} = u(z) + \overline{x}$, where $z = \bar{t}$. The equation of motion can be reduced to $u'' + \mu u' = 0$, where $u' = \frac{du}{dz}$ .Then the exact solution is obtained as follow:

$$\overline{w} = \overline{x} + b_1 e^{-\mu \bar{t}} + b_2, \tag{13}$$

where b_1, b_2 are arbitrary constants.

5. Concluding Remarks

Lie symmetry analysis is applied to the quintic nonlinear equation of motion. The numbers of the independent variables are reduced by one by using infinitesimal generators. Four linearly independent generators are classified based on the optimal system and then are used to reduce the partial differential equation to the ordinary differential equation. In some cases, the ordinary differential equations can be solved analytically and exact solutions are obtained.In this work, some cases of the ordinary differential equations which are reduced from the partial differential equation cannot be solved analytically. In order to solve these equations, we may apply the idea of Meleshko's procedure to reduce the order of the equation from the fourth order ordinary differential equations to the third and the second order, respectively. After that, the exact solutions of the reducing order ordinary differential equations may be given.

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